Positivity relations and localic suplattices

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CORCON, 24-27 March 2014, Genova
Subject: topological closure and related notions.

Framework: constructive point-free Topology
(Formal Topology / the theory of Locales).

Aim: to uncover the links between
- positivity relations (in Formal Topology) and
- lower powerlocales / localic suplattices (in Locale theory).

[Here we do not bother much about the issue of predicativity VS impredicativity.]
Overt weakly closed sublocales


Each overt weakly closed sublocale of \( L \) is generated as follows:

1. fix a suplattice \( a \)
2. impose the EXTRA condition \( a \leq \bigvee \{ \phi(x) \mid x \in \{a\} \} \) for every \( a \in L \).

**IDEA:** \( \phi \) becomes a "positivity predicate".

\( a = \) complete join-semilattice
\( b = \) join-preserving map

CLASSICALLY: you declare \( a = 0 \) whenever \( \phi(a) \) is false.

This is just a closed sublocale.
Overt weakly closed sublocales


Each **overt weakly closed** sublocale of $L$ is **generated** as follows:

1. fix a suplattice\(^a\) homomorphism\(^b\) $\varphi : L \to \Omega = Pow(1) = \text{“truth values”}$;
2. impose the EXTRA condition $a \leq \bigvee \{x \in \{a\} \mid \varphi(x)\}$ for every $a \in L$.

IDEA: $\varphi$ becomes a **“positivity predicate”**.

---

\(^a\) = complete join-semilattice
\(^b\) = join-preserving map

CLASSICALLY: you declare $a = 0$ whenever $\varphi(a)$ is false.
This is just a **closed** sublocale.
Overt weakly closed sublocales are equivalent to... 


Overt weakly closed sublocales of $L \cong \text{SupLat}(L, \Omega) \cong$ points of the lower powerlocale $\mathcal{PL}$
$\cong$ splitting subsets of $L$.

Lower powerlocale: localic version of the *lower Vietoris hyperspace*.\(^1\)

Splitting subsets: $Z \subseteq L$ s.t.

\[
\frac{a \leq \bigvee X \quad a \in Z}{X \nsubseteq Z} \quad (\text{for every } X \subseteq L).\(^2\)
\]

---

\(^1\) = topology on the closed sets with $\{\{C \text{ closed} \mid C \nsubseteq A\} \mid A \text{ open}\}$ as a subbase.

\(^2\) Here $X \nsubseteq Z$ means “$X \cap Z$ is inhabited”. 
Overt weakly closed sublocales are equivalent to...


Overt weakly closed sublocales of $L \simeq \text{SupLat}(L, \Omega)$

points of the lower powerlocale $\mathcal{P}L$

splitting subsets of $L$.

Lower powerlocale: localic version of the *lower Vietoris hyperspace*.$^1$

Splitting subsets: $Z \subseteq L$ s.t. $a \leq \bigvee X \Rightarrow a \in Z$ (for every $X \subseteq L$).$^2$

These appear in Formal Topology as formal closed subset w.r.t. a suitable positivity relation.

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Formal Closed subsets and Positivity relations
in Formal Topology

Let $L$ be “presented” via a **base** $S \subseteq L$ and a **cover relation** $\triangleleft \subseteq S \times \text{Pow}(S)$.

- **Impredicatively:** always possible - take $S = L$ and $a \triangleleft U$ iff $a \leq \bigvee U$
- **Predicatively:** this is the only way to deal with locales!
Formal Closed subsets and Positivity relations in Formal Topology

Let $L$ be “presented” via a **base** $S \subseteq L$ and a **cover relation** $\triangleleft \subseteq S \times \mathcal{P}(S)$.

- Impredicatively: always possible - take $S = L$ and $a \triangleleft U$ iff $a \leq \bigvee U$
- Predicatively: this is the only way to deal with locales!

**Positivity relation** $\ltimes \subseteq S \times \mathcal{P}(S)$ (compatible with a given cover relation)

\[
\begin{align*}
\frac{a \ltimes U}{a \in U} & \quad + \quad \frac{a \ltimes U \quad U \subseteq V}{a \ltimes V} & \quad + \quad \frac{a \ltimes U}{a \ltimes \{b \in S \mid b \ltimes U\}} & \quad + \quad \frac{a \triangleleft U \quad a \ltimes V}{(\exists u \in U)(u \ltimes V)}
\end{align*}
\]

\[\{x \in S \mid x \ltimes U\} \leftarrow \text{formal closed subset} \text{ (indexed by } U \subseteq S).\]

It’s a splitting subset.
Positivity relations and splitting subsets

\[ \text{Split}(L) = \{\text{splitting subsets of } L\} \text{ is a suplattice (w.r.t. union).} \]

[This doesn’t mean that join of closed is closed!]
Positivity relations and splitting subsets

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Positivity relation on \( L \) = sub-suplattice of \( \text{Split}(L) \)

\[ \text{FormalClosed}(\times) \overset{\text{def}}{=} \{\text{formal closed subsets w.r.t. } \times\} \text{ is a sub-suplattice of } \text{Split}(L). \]
Positivity relations and splitting subsets

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Positivity relation on $L = \text{sub-suplattice of } \text{Split}(L)$

$\text{FormalClosed}(\ltimes) \overset{\text{def}}{=} \{\text{formal closed subsets w.r.t. } \ltimes\}$ is a sub-suplattice of $\text{Split}(L)$.

Vice versa: if $M \hookrightarrow \text{Split}(L)$, then

$$a \ltimes_M U \overset{\text{def}}{\iff} (\exists X \in M)(a \in X \subseteq U)$$

is a positivity relation on $L$ s.t. $\text{FormalClosed}(\ltimes_M) = M$. 
Positivity relations and splitting subsets

\( \text{Split}(L) = \{ \text{splitting subsets of } L \} \) is a \textbf{suplattice} (w.r.t. union).

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Positivity relation on \( L = \text{sub-suplattice of } \text{Split}(L) \)

\[ \text{FormalClosed}(\ltimes) \overset{\text{def}}{=} \{ \text{formal closed subsets w.r.t. } \ltimes \} \text{ is a sub-suplattice of } \text{Split}(L). \]

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This works for any presentation \((S, \ltimes)\) of \( L \), that is,

the suplattice \( \text{FormalClosed}(\ltimes) \) is \textbf{independent} from the presentation of \( L \).
Positivity on a locale
in terms of suplattice homomorphisms

Positive relation on $L$ \[\cong\] any sub-suplattice of $\text{Split}(L)$
\[\cong\] any sub-suplattice of $\text{SupLat}(L, \Omega)$.

[Because $\text{Split}(L) \cong \text{SupLat}(L, \Omega)$.]
Positivity on a locale
in terms of suplattice homomorphisms

Positivity relation on $L$ $\iff$ any sub-suplattice of $\text{Split}(L)$
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[Because $\text{Split}(L) \cong \text{SupLat}(L, \Omega)$.

With classical logic. . .

$\text{SupLat}(L, \Omega) \cong \text{SupLat}(\Omega^{op}, L^{op}) \cong \text{SupLat}(\Omega, L^{op}) \cong L^{op}$

Therefore, a positivity relation $X$ on $L$ is

- any sub-suplattice $X \hookrightarrow L^{op}$
- any suplattice quotient $L \twoheadrightarrow Y$ (with $Y = X^{op}$)
- any suplattice of the form $\text{SupLat}(Y, \Omega)$ for some suplattice quotient $L \twoheadrightarrow Y$. 
What is $\mathcal{P}L$, the lower powerlocale of $L$?

It’s underlying frame is the **free** frame generated by $L$ qua suplattice.
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Corollary: $\text{Points}(\mathcal{P}L) \cong$ frame homomorphisms $\mathcal{P}L \to \Omega \cong \text{SupLat}(L, \Omega)$. 
Positivity relations and the lower powerlocale

What is \( \mathcal{P}L \), the lower powerlocale of \( L \)?

It’s underlying frame is the free frame generated by \( L \) qua suplattice.

Corollary: Points(\( \mathcal{P}L \)) \( \cong \) frame homomorphisms \( \mathcal{P}L \to \Omega \cong \text{SupLat}(L, \Omega) \).

Therefore: a positivity relation on \( L \) \( \cong \) any sub-suplattice of \( \text{SupLat}(L, \Omega) \)
\( \cong \) any sub-suplattice of Points(\( \mathcal{P}L \))
(w.r.t. specialization order).
What is $\mathcal{P}L$, the lower powerlocale of $L$?

It’s underlying frame is the free frame generated by $L$ qua suplattice.

**Corollary:** $\text{Points}(\mathcal{P}L) \cong \text{frame homomorphisms } \mathcal{P}L \to \Omega \cong \text{SupLat}(L, \Omega)$.

Therefore: a positivity relation on $L \cong$ any sub-suplattice of $\text{SupLat}(L, \Omega)$

$\cong$ any sub-suplattice of $\text{Points}(\mathcal{P}L)$

(w.r.t. specialization order).

**Question:**

is a positivity relation a mere suplattice of points of $\mathcal{P}L$?

Or maybe it’s the points of a sublocale of $\mathcal{P}L$?
Localic suplattices


\( \mathcal{P} \) is (the functor part of) a monad on the category of locales.

A **localic suplattice** \( \overset{\text{def}}{=} \) an algebra \( \mathcal{P}X \to X \) for this monad.
Localic suplattices


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A **localic suplattice** \( \overset{\text{def}}{=} \) an algebra \( \mathcal{P}X \to X \) for this monad.

1. \( X \) localic suplattice \( \implies \) \( \text{Pt}(X) \) suplattice (w.r.t. specialization order);
2. \( X \) localic **sub**-suplattice of \( \mathcal{P}L \implies \text{Pt}(X) \) positivity relation on \( L \).

(CLASS)

Every positivity relation on \( L \) is of the form \( \text{Pt}(X) \) for some localic sub-suplattice \( X \) of \( \mathcal{P}L \).
Examples: positivity relations on a topological space $X$

For $p$ a point let $\mathcal{N}p \overset{\text{def}}{=} \text{open neighbourhoods of } p$.

Then $a \times_X U \overset{\text{def}}{=} (\exists p \in X)(a \in \mathcal{N}p \subseteq U)$ is a positivity relation on $X$ and $Formal\text{Closed}(\times_X) \cong \{\text{closed}^3 \text{ subsets of } X\}$

---

3Here “closed” means “contains its own adherent points” (constructively not the same thing as “is the complement of an open set”).
Examples: positivity relations on a topological space $X$

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Then $\overset{\text{def}}{\iff} (\exists p \in X)(a \in \mathcal{N}p \subseteq U)$ is a positivity relation on $X$ and

$$\text{FormalClosed}(\ltimes_X) \cong \{\text{closed}^3 \text{ subsets of } X\}$$

More generally . . .

for every subspace $Y \subseteq X$,

$$\{\text{closed sets in the subspace } Y\}$$

 corresponds to a positivity relation on $X$.

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Morphisms that respect positivity

Positive topology $\overset{\text{def}}{=} (S, \triangleleft, \asymp) \cong (L, \Phi)$
(a.k.a. Balanced formal topology)

where $\Phi \hookrightarrow \text{SupLat}(L, \Omega)$
Morphisms that respect positivity

Positive topology $\overset{\text{def}}{=} (S, \triangledown, \bowtie) \cong (L, \Phi)$

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where $\Phi \hookrightarrow \text{SupLat}(L, \Omega)$

A morphism $(L_1, \Phi_1) \rightarrow (L_2, \Phi_2)$ is...

- an arrow $L_1 \xrightarrow{f} L_2$ of locales (i.e. a homomorphism $L_2 \xrightarrow{f^{-}} L_1$ of frames)
- such that $L_1 \xrightarrow{\varphi} \Omega$ in $\Phi_1 \implies L_2 \xrightarrow{f^{-}} L_1 \xrightarrow{\varphi} \Omega$ in $\Phi_2$

(the “inverse image” of a formal closed subset is a formal closed subset).
Morphisms that respect positivity

Positive topology \( \overset{\text{def}}{=} (S, \triangleleft, \triangleright) \cong (L, \Phi) \)
(a.k.a. Balanced formal topology)

\( \downarrow \text{impredicatively} \)

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- such that \(L_1 \xrightarrow{\varphi} \Omega \text{ in } \Phi_1 \implies L_2 \xrightarrow{f^-} L_1 \xrightarrow{\varphi} \Omega \text{ in } \Phi_2\)

(the “inverse image” of a formal closed subset is a formal closed subset).

\[ \text{SupLat}(L_1, \Omega) \xrightarrow{(\_\_\_)\circ f^-} \text{SupLat}(L_2, \Omega) \]

\[ \Phi_1 \overset{\downarrow}{\longrightarrow} \Phi_2 \]
The greatest positivity relation

Every locale $L$ has a greatest positivity relation:

$\bowtie_{\text{max}} \cong \Phi_{\text{max}} \overset{\text{def}}{=} \text{SupLat}(L, \Omega) \cong \{\text{all overt weakly closed sublocales of } L\}$

[Actually, positivity relations on $L$ form a SUPLATTICE, the suplattice of sub-suplattices of $\text{SupLat}(L, \Omega)$.]
The greatest positivity relation

Every locale $L$ has a greatest positivity relation:

\[ \ll_{\max} \cong \phi_{\max} \overset{\text{def}}{=} \text{SupLat}(L, \Omega) \cong \{ \text{all overt weakly closed sublocales of } L \} \]

[Actually, positivity relations on $L$ form a SUPLATTICE, the suplattice of sub-suplattices of $\text{SupLat}(L, \Omega)$.]

Constructively . . .

if $L = (S, \triangleleft)$ is inductively generated, then $\ll_{\max}$ is generated by co-induction.

(Martin-Löf & Sambin - Generating Positivity by Coinduction)

See also:
Embedding locales into locales-with-positivity

\( \text{Loc} = \) category of locales
\( \text{PLoc} = \) category of locales-with-positivity (and morphisms that respect positivity)
  (impredicative version of \( \text{PTop} \), category of positive topologies).
Embedding locales into locales-with-positivity

**Loc** = category of locales

**PLoc** = category of locales-with-positivity (and morphisms that respect positivity)

(impredicative version of **PTop**, category of positive topologies).

By definition of morphism that respect positivity:

an arrow \((L, \Phi) \rightarrow (L', \text{SupLat}(L', \Omega))\) in **PLoc**

is equivalent to

an arrow \(L \rightarrow L'\) in **Loc**.
Embedding locales into locales-with-positivity

\( \textbf{Loc} \) = category of locales

\( \textbf{PLoc} \) = category of locales-with-positivity (and morphisms that respect positivity)

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an arrow \((L, \Phi) \rightarrow ((L', \textbf{SupLat}(L', \Omega))\) in \( \textbf{PLoc} \)

is equivalent to

an arrow \(L \rightarrow L'\) in \( \textbf{Loc} \).

\[
\begin{array}{ccc}
\textbf{Loc} & \xrightarrow{\mathcal{I}} & \textbf{PLoc} \\
\downarrow & & \downarrow \\
\mathcal{U} & & \\
\end{array}
\]

where \( \mathcal{U} \) “forgets” positivity relations and \( \mathcal{I} \) adds the greatest positivity relations.
Embedding locales into locales-with-positivity

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By definition of morphism that respect positivity:

\[
\text{an arrow } (L, \Phi) \rightarrow ((L', \text{SupLat}(L', \Omega))) \text{ in } \text{PLoc}
\]

\[
\text{is equivalent to}
\]

\[
\text{an arrow } L \rightarrow L' \text{ in } \text{Loc}.
\]

\[
\text{Loc} \xrightarrow{I} \xleftarrow{U} \text{PLoc}
\]

where \( U \) “forgets” positivity relations and \( I \) adds the greatest positivity relations.

\( \text{Loc} \) is equivalent to a \textbf{reflective} subcategory of \( \text{PLoc} \).
Points of a positive locale

1 = terminal object of PLoc = terminal locale + greatest positivity

\[
\begin{align*}
\text{Points}(L, \Phi) &= \text{PLoc}(1, (L, \Phi)) \\
&= \text{Frame}(L, \Omega) \cap \Phi \\
&= \{ \varphi \in \Phi \mid \varphi \text{ preserves finite meets} \} \\
&= \text{Points}(L) \cap \Phi \\
&\text{(with some abuse of notation)}
\end{align*}
\]

Idea: a positivity is a way for selecting points.


Note that Points(L) = Points(L, \Phi_{max}).
The positivity relation induced by a sub-locale

If \( M \hookrightarrow L \) is a sub-locale, then

\[
\text{SupLat}(M, \Omega)
\]

is a positivity relation on \( L \) and

\[
\text{Points}(L, \text{SupLat}(M, \Omega)) = \text{Points}(L) \cap \text{SupLat}(M, \Omega) = \text{Points}(M).
\]

So there are two ways of dealing with a (sober) sub-space:

1. either define a sublocale
2. or use a positivity relation.

[CLASS: they are the same.]
The adjunction between $\textbf{Top}$ and $\textbf{PLoc}$

Let $\textbf{Top}$ be the category of topological spaces. There exists an adjunction

$\textbf{Top}$

$\rightarrow$

$\textbf{PLoc}$

$X \mapsto (\Omega X, \ltimes X)$

$\text{Points}(L, \Phi) \leftrightarrow (L, \Phi)$

[Recall: $\ltimes X$ corresponds to all closed subsets of $X$ (in the sense of adherence points).]
The adjunction between \textbf{Top} and \textbf{PLoc}

Let \textbf{Top} be the category of topological spaces. There exists an adjunction

\[
\begin{array}{ccc}
\text{Top} & \xrightarrow{\Omega} & \text{PLoc} \\
X & \mapsto & (\Omega X, \star_X) \\
\text{Points}(L, \Phi) & \leftrightarrow & (L, \Phi)
\end{array}
\]

[Recall: \(\star_X\) corresponds to all closed subsets of \(X\) (in the sense of adherence points).]

Constructively this is NOT an extension of the usual adjunction between \textbf{Top} and \textbf{Loc} because

\[
\begin{array}{ccc}
\text{Top} & \xrightarrow{\Omega} & \text{Loc} \\
\text{Points} & \xleftarrow{U} & \text{PLoc}
\end{array}
\]

Their composition is not an adjunction!
Two notions of sobriety
for a topological space $X$

points $= \text{Points}(\Omega X, \times_{\text{max}}) = \text{Points}(\Omega X)$

“strong” points $= \text{Points}(\Omega X, \times_X) = \text{Points}(\Omega X) \cap \times_X$

$\alpha$ is “strong” if for all $a \in \alpha$, there exists $p \in X$ s.t. $a \in Np \subseteq \alpha$.

sober $X = \text{Points}(\Omega X, \times_{\text{max}})$

weakly sober $X = \text{Points}(\Omega X, \times_X)$
Two notions of sobriety
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sober $X = \text{Points}(\Omega X, \sqcap_{\text{max}})$

weakly sober $X = \text{Points}(\Omega X, \sqcap_X)$

If $X$ is $T_2$, then $X$ is weakly sober.

On the contrary, if “$T_2 \Rightarrow \text{sober}$” were true, then LPO would hold.

Fourman & Scott, *Sheaves and Logic*, in *Applications of sheaves*, LNM 753 (1979)

Positivity relations on suplattices

The notion of a positivity relation makes sense also for the category $\text{SupLat}$ of suplattices.

\[
\text{basic topology} = \text{suplattice } L + \text{ positivity } \Phi \hookrightarrow \text{SupLat}(L, \Omega)
\]

As before, every suplattice $L$ can be identified with the basic topology $(L, \Phi_{max})$. 
Conclusions and Future work

Summing up: every locale $L$ can be equipped with several positivity relations each of which corresponds to a constraint on points

1. every sublocale $M \hookrightarrow L$ gives the positivity $\text{SupLat}(M, \Omega)$;
2. every localic suplattice $X \hookrightarrow \mathcal{P}L$ gives the positivity $\text{Points}(X)$;
3. classically, (1 and) 2 give all possible positivity relations on $L$;
4. constructively, positivity relations could be more expressive than 1 and 2.

Work must go on!

Is 4 above “really” more expressive than 1 and 2? (What kind of restrictions on points can be obtained “only” by means of a positivity relation and not as a sublocale?)

Does the category $\text{PLoc}$ (of locales with positivity relations) satisfy the same/more/less categorical properties than $\text{Loc}$?

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- Is 4 above “really” more expressive than 1 and 2?
- (What kind of restrictions on points can be obtained “only” by means of a positivity relation and not as a sublocale?)
- Does the category $\mathbf{PLoc}$ (of locales with positivity relations) satisfy the same/more/less categorical properties than $\mathbf{Loc}$?
- . . .


References


Thank you!