Point-free characterisation of Bishop compact metric spaces

Tatsuji Kawai
tatsuji.kawai@jaist.ac.jp

School of Information Science
Japan Advanced Institute of Science and Technology

Topology in Constructive mathematics

Metric space

The theory of metric spaces as described in Bishop's *Foundations of Constructive Analysis* is well established in constructive mathematics. However, its extension to general topology has a major difficulty.

- Most of the compact metric spaces fail to be topologically compact without recourse to Fan theorem.

Point-free topology

A promising approach to general topology (without relying on Fan theorem) is **formal topology** (Sambin, 1987).

- A point-free topology adapted from the theory of locale (frame).
- Many spaces behave better in formal topology. Formal Cantor space and Formal unit interval are topologically compact.
- Successfully constructivised many results of classical topology: Tychonoff's theorem for compact topologies (without choice).

Connection between metric space and formal topology

The connection between Bishop's metric space and formal topology has been unclear until recently.

- The compactness in formal topology via open cover and compactness in Bishop's metric space via completeness and totally boundedness.

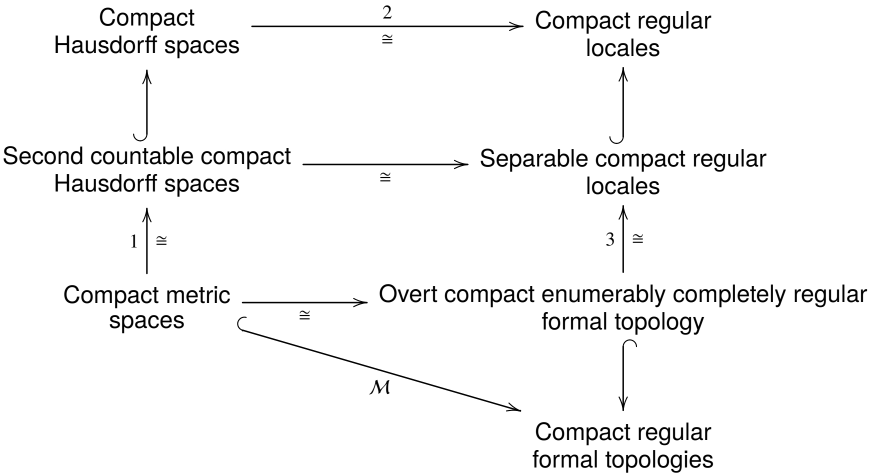
Theorem (Palmgren, 2007). There exists a full and faithful functor, called a localic completion, from the category of locally compact metric spaces to the category of locally compact regular formal topologies.

Our work characterises the image of the compact metric spaces under the localic completion – point-free characterisation of compact metric spaces.

Theorem. Let $\mathcal{S}$ be a formal topology. Then, the following are equivalent:

1. $\mathcal{S}$ is isomorphic to an overt compact enumerably completely regular formal topology.
2. $\mathcal{S}$ is isomorphic to a compact overt subtopology of $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$.
3. $\mathcal{S}$ is isomorphic to a localic completion of some compact metric space.

Note. We work in Bishop style constructive mathematics: we use some choice principles, i.e. Countable Choice ($\mathbf{AC}_{\omega}$) and Dependent Choice (**DC**).



1. Fan theorem
2. Classical logic + Prime Ideal theorem
3. Classical logic

Formal topology

A **formal topology** $\mathcal{S}$ is a triple $\mathcal{S} = (S, \lhd, \leq)$ where $(S, \leq)$ is a preorder and $\lhd \subseteq S \times \mathsf{Pow}(S)$ is a **covering relation** on S such that

$$\frac{a \in U}{a \lhd U}, \quad \frac{a \leq b}{a \lhd b}, \quad \frac{a \lhd U \quad U \lhd V}{a \lhd V}, \quad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V},$$

for all $a, b \in S$ and $U, V \subseteq S$ where

$$U \lhd V \overset{\text{def}}{\iff} (\forall a \in U)\, a \lhd V,$$

$$U \downarrow V \overset{\text{def}}{=} {\downarrow}U \cap {\downarrow}V = \{c \in S \mid (\exists a \in U)\,(\exists b \in V)\, c \leq a \;\&\; c \leq b\}.$$

Formal topology

Let $\mathcal{S} = (S, \lhd, \leq)$ be a formal topology. The operator

$$\mathcal{A} : \mathsf{Pow}(S) \to \mathsf{Pow}(S) : U \mapsto \mathcal{A}\, U = \{a \in S \mid a \lhd U\}$$

is an order theoretic closure operation on $\mathsf{Pow}(S)$. The collection of its fixed points, denoted by $Sat(\mathcal{S})$, forms a **frame** (or a complete Heyting algebra), a complete lattice where finite meets distribute over arbitrary joins:

$$\mathcal{A}\, U \wedge \bigvee_{i \in I} \mathcal{A}\, U_i = \bigvee_{i \in I} \mathcal{A}\, U \wedge \mathcal{A}\, U_i$$

for all $U \subseteq S$ and a family of subsets $U_i \subseteq S$ $(i \in I)$.

Formal topology map

Let $\mathcal{S}$ and $\mathcal{S}'$ be formal topologies. A **(formal topology) map** from $\mathcal{S}$ to $\mathcal{S}'$ is a relation $r \subseteq S \times S'$ such that

1. $S \lhd r^- S'$,
2. $r^-\{a\} \downarrow r^-\{b\} \lhd r^-(\{a\} \downarrow' \{b\})$,
3. $a \lhd' U \implies r^- \{a\} \lhd r^- U$

for all $a, b \in S'$ and $U \subseteq S'$. The collection $Hom(\mathcal{S}, \mathcal{S}')$ of formal topology maps is equipped with the equality $r = s \overset{\text{def}}{\iff} \mathcal{A}\, r^-\{a\} = \mathcal{A}\, s^-\{a\}$ $(a \in S')$. A formal topology map $r : \mathcal{S} \to \mathcal{S}'$ induces a frame map

$$\mathcal{A}\, r^-(-) : Sat(\mathcal{S}') \to Sat(\mathcal{S}).$$

A **point** of a formal topology $\mathcal{S}$ is a subset $\alpha \subseteq S$ such that

1. $(\exists a \in S)\, a \in \alpha$,
2. $a, b \in \alpha \implies (c \in a \downarrow b)\, c \in \alpha$,
3. $a \lhd U \;\&\; a \in \alpha \implies (\exists b \in U)\, b \in \alpha$.

The collection of points of $\mathcal{S}$ is denoted by $Pt(\mathcal{S})$.

Inductively generated formal topology

Let S be a set. An **axiom-set** on S is a pair (I, C), where $(I(a))_{a\in S}$ is a family of sets, and C is a family $(C(a,i))_{a\in S, i\in I(a)}$ of subsets of S.

Theorem (Coquand, Sambin, Smith, and Valentini, 2003). Let $(S, \leq)$ be a preordered set, and let (I, C) be an axiom-set on S. Then, there exists a covering relation $\lhd_{I,C}$ inductively generated by the following rules:

$$\frac{a \in U}{a \lhd_{I,C} U} \text{ (reflexivity)}, \quad \frac{a \leq b \quad b \lhd_{I,C} U}{a \lhd_{I,C} U} \text{ ($\leq$-left)},$$

$$\frac{a \leq b \quad i \in I(b) \quad a \downarrow C(b,i) \lhd_{I,C} U}{a \lhd_{I,C} U} \text{ ($\leq$-infinity)}.$$

The relation $\lhd_{I,C}$ is the least covering relation on S which satisfies ($\leq$-left) and $a \lhd_{I,C} C(a,i)$ for each $a \in S$ and $i \in I(a)$.

The formal topology $\mathcal{S} = (S, \lhd_{I,C}, \leq)$ together with the axiom set (I, C) which generates $\lhd_{I,C}$ is called an **inductively generated formal topology**. A pair $(a, C(a,i))$ for each $a \in S$ and $i \in I(a)$ is called an axiom of $\mathcal{S}$ and will be written $a \lhd_{I,C} C(a,i)$.

Formal Cantor space Let $S = \{0,1\}^*$ be ordered by $l \leq l' \overset{\text{def}}{\iff} (\exists k \in S)\, l' * k = l$. Formal Cantor space $\mathcal{C}$ is generated by the following axiom-set on S:

$$l \lhd \{l * \langle 0 \rangle,\ l * \langle 1 \rangle\}$$

Explicitly, define $I(l) = \{*\}$ and $C(l,*) = \{l * \langle 0 \rangle, l * \langle 1 \rangle\}$ for each $l \in S$. We have $Pt(\mathcal{C}) \cong 2^{\mathbb{N}}$.

Formal Reals Let $S_{\mathcal{R}} = \{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\}$ be ordered by $(r,s) \leq (p,q) \overset{\text{def}}{\iff} p \leq r \;\&\; s \leq q$. Formal reals $\mathcal{R}$ is generated by the following axiom set on S.

(R1) $(p,q) \lhd_{\mathcal{R}} \{(r,s) \in S_{\mathcal{R}} \mid p < r < s < q\}$,

(R2) $(p,q) \lhd_{\mathcal{R}} \{(p,s),(r,q)\}$ for each $p < r < s < q$.

We have $Pt(\mathcal{R}) \cong \mathbb{R}$, where $\mathbb{R}$ is the Dedekind cuts.

Subtopology

A **subtopology** $\mathcal{S}'$ of a formal topology $\mathcal{S} = (S, \lhd, \leq)$ is a formal topology of the form $\mathcal{S}' = (S, \lhd', \leq)$ such that $\lhd \subseteq \lhd'$ (which implies $Pt(\mathcal{S}') \subseteq Pt(\mathcal{S})$).

The **image** of a map $r : \mathcal{S}' \to \mathcal{S}$ is a subtopology $Im(r) = (S, \lhd_{Im(r)}, \leq)$ of $\mathcal{S}$ where $a \lhd_{Im(r)} U \iff r^{-}\{a\} \lhd' r^{-}U$.

A formal topology map $r : \mathcal{S}' \to \mathcal{S}$ is an **embedding** if $\mathcal{A}\, r^{-}(-) : Sat(\mathcal{S}) \to Sat(\mathcal{S}')$ is surjective (iff r is impredicatively a regular monomorphim.) We have $Im(r) \cong \mathcal{S}'$.

Example. Given a subset $V \subseteq S$ of a formal topology $\mathcal{S}$, the cover $\lhd^{\mathcal{S}-V}$ on S defined by

$$a \lhd^{\mathcal{S}-V} U \overset{\text{def}}{\iff} a \lhd V \cup U$$

forms a subtopology $(S, \lhd^{\mathcal{S}-V}, \leq)$ of $\mathcal{S}$. A subtopology of $\mathcal{S}$ with a cover of the form $\lhd^{\mathcal{S}-V}$ is called a **closed** subtopology of $\mathcal{S}$, which we denote by $\mathcal{S}^{\mathcal{S}-V}$.

The formal unit interval $\mathcal{I}[0,1]$ is a closed subtopology of the formal reals $\mathcal{R}$ determined by $V = \{(p,q) \in S_{\mathcal{R}} \mid q \leq 0 \vee 1 \leq p\}$, i.e.

$$\mathcal{I}[0,1] = \mathcal{R}^{\mathcal{R}-V}.$$

Overt formal topology

Let $\mathcal{S}$ be a formal topology. A subset $Pos \subseteq S$ is called **splitting** if

$$\textbf{(Mon)} \quad a \lhd U \;\&\; Pos(a) \implies (\exists b \in U)\, Pos(b).$$

A splitting subset $Pos \subseteq S$ is called a **positivity predicate** on $\mathcal{S}$ if

$$\textbf{(Pos)} \quad a \lhd \{x \in S \mid x = a \;\&\; Pos(a)\}.$$

Intuitively, $Pos(a)$ if "the basic open a is inhabited". Every formal topology admits at most one positivity predicate. A formal topology is **overt** if it is equipped with a positivity predicate.

Lemma. The image of an overt formal topology $\mathcal{S}' = (S', \lhd', \leq', Pos)$ under a map $r : \mathcal{S}' \to \mathcal{S}$ is overt with the positivity $rPos = \{a \in S \mid (\exists b \in Pos)\, b\, r\, a\}$.

Localic completion (Vickers, 2005; Palmgren, 2007)

Let $X = (X, \rho)$ be a metric space, and let $\mathbb{Q}^{>0}$ be the set of positive rationals. A **formal ball** $\mathsf{b}(x, \varepsilon)$ is a pair $(x, \varepsilon) \in X \times \mathbb{Q}^{>0}$. We write M_X for $X \times \mathbb{Q}^{>0}$. Define an order $\leq_X$ and a strict order $<_X$ on M_X by

$$\mathsf{b}(x, \delta) \leq_X \mathsf{b}(y, \varepsilon) \overset{\text{def}}{\iff} \rho(x, y) + \delta \leq \varepsilon,$$

$$\mathsf{b}(x, \delta) <_X \mathsf{b}(y, \varepsilon) \overset{\text{def}}{\iff} \rho(x, y) + \delta < \varepsilon.$$

Note The conditions are not equivalent to the (strict) inclusion between the actual balls $B(x, \varepsilon) = \{y \in X \mid \rho(x, y) < \varepsilon\}$.

The **localic completion** of a metric space (X, ρ) is a formal topology $\mathcal{M}(X) = (M_X, \lhd_X, \leq_X)$ inductively generated by the following axiom-set on M_X:

(M1) $a \lhd_X \{b \in M_X \mid b <_X a\}$,

(M2) $a \lhd_X C_\varepsilon$ for each $\varepsilon \in \mathbb{Q}^{>0}$

for all $a \in M_X$, where we define $C_\varepsilon = \{\mathsf{b}(x, \varepsilon) \in M_X \mid x \in X\}$, the set of formal balls with radius ε.

Localic completion

For any metric space $X = (X, \rho)$

- its localic completion $\mathcal{M}(X)$ is always overt.
- the points $Pt(\mathcal{M}(X))$ is a completion of X: $Pt(\mathcal{M}(X))$ is isometric to the set $\tilde{X}$ of Cauchy sequences on X modulo the standard equality.
- if $Y \subseteq X$ is a dense subset of X, then $\mathcal{M}(Y) \cong \mathcal{M}(X)$.
- $\mathcal{M}(2^{\mathbb{N}}) \cong \mathcal{C}$, $\mathcal{M}(\mathbb{Q}) \cong \mathcal{M}(\mathbb{R}) \cong \mathcal{R}$ and $\mathcal{M}([0,1]) \cong \mathcal{I}[0,1]$.

A metric space is **compact** if it is complete and totally bounded.

A formal topology $\mathcal{S}$ is **compact** if

$$S \lhd U \implies (\exists U_0 \in \mathrm{Fin}(U))\, S \lhd U_0$$

for all $U \subseteq S$.

Theorem (Palmgren, 2007). The localic completion $\mathcal{M}$ restricts to a full and faithful functor $\mathcal{M} : \textbf{Comp} \to \textbf{KRFTop}$, where

Comp the category of compact metric spaces and uniformly continuous functions.

KRFTop the category of compact regular formal topologies and maps.

Spitters (2010) and Coquand, Palmgren, and Spitters (2011) observed that a compact subspace of a Bishop locally compact metric space gives rise to a compact overt subtopologies of its localic completion, and vise versa.

Regular formal topology

Let $\mathcal{S}$ be a formal topology, and $U, V \subseteq S$. Define

$$U \lll V \overset{\text{def}}{\iff} S \lhd U^* \cup V$$

where $U^* = \{a \in S \mid a \downarrow U \lhd \emptyset\}$. A formal topology $\mathcal{S}$ is **regular** if there exists a function $wc \colon S \to \mathrm{Pow}(S)$ such that for all $a \in S$

- $(\forall b \in wc(a))\, \{b\} \lll \{a\}$,
- $a \lhd wc(a)$.

Lemma. Let $\mathcal{S}'$ be a subtopology of a compact regular formal topology. Then

- $\mathcal{S}'$ is compact iff it is closed.
- $\mathcal{S}'$ is compact overt (or overt compact) iff it is overt closed.

Lemma. The localic completion $\mathcal{M}(X)$ of a metric space X is regular.

Proof. For each $a = \mathrm{b}(x, \varepsilon)$, we have $a \lhd_X \{b \in M_X \mid b <_X a\}$. Define $wc(a) = \{b \in M_X \mid b <_X a\}$. We can even take $wc(a) = \{\mathrm{b}(x, \delta) \in M_X \mid \delta <_X \varepsilon\}$, since $\mathrm{b}(x, \varepsilon) \lhd_X \{\mathrm{b}(x, \delta) \in M_X \mid \delta <_X \varepsilon\}$ is derivable in $\mathcal{M}(X)$. □

Located subtopology

Definition (Spitters (2010)). Let $\mathcal{S}$ be a compact regular formal topology with the function $wc : S \to \mathsf{Pow}(S)$. A subset $Pos \subseteq S$ is called a **located** predicate on $\mathcal{S}$ if Pos is splitting and satisfies

$$b \in wc(a) \implies b \in \neg Pos \vee a \in Pos$$

for all $a, b \in S$, where $\neg Pos = \{a \in S \mid \neg(a \in Pos)\}$. A subtopology $\mathcal{S}'$ of a compact regular formal topology $\mathcal{S}$ is **located** if there exists a located predicate Pos on $\mathcal{S}$ such that $\mathcal{S}' = \mathcal{S}^{S-\neg Pos}$, i.e. $\mathcal{S}'$ coincides with the closed subtopology determined by $\neg Pos$.

Lemma. Let $\mathcal{S}$ be a compact regular formal topology, and let $\mathcal{S}'$ be a subtopology of $\mathcal{S}$. Then the following are equivalent:

1. $\mathcal{S}'$ is located.
2. $\mathcal{S}'$ is overt closed.
3. $\mathcal{S}'$ is compact overt.

Compact overt sub-topologies of a localic completion

Let $X = (X, \rho)$ be a compact metric space, and let $Y \subseteq X$ be a compact subspace of X. Then, $\Diamond Y = \{\mathrm{b}(x, \varepsilon) \in M_X \mid (\exists y \in Y)\, \rho(x, y) < \varepsilon\}$ is a located subset of $\mathcal{M}(X)$.
Conversely, if $Pos \subseteq M_X$ is a located subset of $\mathcal{M}(X)$, then $Pt(\mathcal{M}(X)^{\mathcal{M}(X)-\neg Pos}) \subseteq Pt(\mathcal{M}(X)) \cong X$ is a compact subspace of X.

Theorem (DC). Let $X = (X, \rho)$ be a compact metric space. Then, up to isomorphism, the localic completion $\mathcal{M}$ induces an order isomorphism between the compact subspaces of X and the compact overt subtopologies of $\mathcal{M}(X)$.

Proof. Given a compact subspace $Y \subseteq X$, its localic completion $\mathcal{M}(Y)$ embeds into $\mathcal{M}(X)$ as an overt compact subtopology determined by $\Diamond Y$. Conversely, given a compact overt subtopology $\mathcal{S}$ of $\mathcal{M}(X)$, the points $Pt(\mathcal{S})$ is metrically isomorphic to a compact subset of X. □

Corollary. The following are equivalent for a formal topology $\mathcal{S}$.

1. $\mathcal{S}$ is isomorphic to $\mathcal{M}(X)$ of some compact metric space X.
2. $\mathcal{S}$ is isomorphic to a compact overt subtopology of $\mathcal{M}(X)$ of some compact metric space X.

The image of countable products

Given a set-indexed family $(\mathcal{S}_i)_{i \in I}$ of inductively generated formal topologies, with an axiom set $(J_i, D_i)_{i \in I}$, its product $\prod_{i \in I} \mathcal{S}_i$ is inductively generated by an axiom-set on $S_\Pi \stackrel{\text{def}}{=} \text{Fin}\left(\sum_{i \in I} S_i\right)$ ordered by

$$A \leq_\Pi B \stackrel{\text{def}}{\iff} (\forall (i,b) \in B)\,(\exists (j,a) \in A)\; i = j \;\&\; a \leq_i b$$

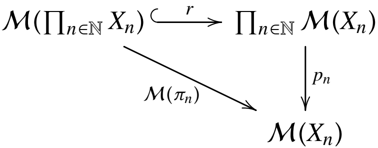
for all $A, B \in S_\Pi$.

(S1) $S_\Pi \lhd_\Pi \{\{(i,a)\} \in S_\Pi \mid a \in S_i\}$ for each $i \in I$,

(S2) $\{(i,a),(i,b)\} \lhd_\Pi \{\{(i,c)\} \in S_\Pi \mid c \leq_i a \;\&\; c \leq_i b\}$,

(S3) $\{(i,a)\} \lhd_\Pi \{\{(i,b)\} \in S_\Pi \mid b \in D_i(a,k)\}$ for each $k \in J_i(a)$.

Proposition (AC_ω). Let $(X_n, \rho_n)_{n \in \mathbb{N}}$ be a sequence of compact metric spaces. The canonical map $r \colon \mathcal{M}(\prod_{n \in \mathbb{N}} X_n) \to \prod_{n \in \mathbb{N}} \mathcal{M}(X_n)$ corresponding to the projections $\mathcal{M}(\pi_n) \colon \mathcal{M}(\prod_{n \in \mathbb{N}} X_n) \to \mathcal{M}(X_n)$ $(n \in \mathbb{N})$ is an embedding. Moreover, the image of $\mathcal{M}(\prod_{n \in \mathbb{N}} X_n)$ in $\prod_{n \in \mathbb{N}} \mathcal{M}(X_n)$ is the largest overt subtopology of $\prod_{n \in \mathbb{N}} \mathcal{M}(X_n)$.



Scale

Let $\mathcal{S}$ be a formal topology, and $U, V \subseteq S$. Define

$$U \lll V \overset{\text{def}}{\iff} S \lhd U^* \cup V$$

where $U^* = \{a \in S \mid a \downarrow U \lhd \emptyset\}$.

Let $\mathbb{I} = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\}$. A **scale** from U to V is a family $\left(U_q\right)_{q \in \mathbb{I}}$ of subsets of S such that

- $U \lhd U_0$, $U_1 \lhd V$,
- $(\forall p, q \in \mathbb{I})\, p < q \implies U_p \lll U_q$.

A scale $\left(U_q\right)_{q \in \mathbb{I}}$ from U to V is **finitary** if $U_q \in \mathrm{Fin}(S)$ for all $q \in \mathbb{I}$. Let

$$U \lll_{\mathsf{Fin}} V \overset{\text{def}}{\iff} \text{there exists a finitary scale from } U \text{ to } V.$$

Proposition (DC). Let $\mathcal{S}$ be a compact regular formal topology. Then, for any $U, V \subseteq S$,

$$U \lll V \implies U \lll_{\mathsf{Fin}} V.$$

Proposition (Johnstone, 1982). Let $\mathcal{S}$ be a formal topology, and let $U, V \subseteq S$. Then, the following are equivalent.

1. There exists a scale from U to V.
2. There exists a formal topology map $r \colon \mathcal{S} \to \mathcal{I}[0,1]$ such that
   - $r^{-}(0,\infty) \downarrow U \lhd \emptyset$,
   - $r^{-}(-\infty,1) \lhd V$.

   where

$$(-\infty,1) \stackrel{\mathsf{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid q = 1\},$$

$$(0,\infty) \stackrel{\mathsf{def}}{=} \{(p,q) \in S_{\mathcal{R}} \mid p = 0\}.$$

Compact enumerably completely regular formal topology

A compact formal topology $\mathcal{S}$ is **enumerably completely regular** if

- there exists a function $wc \colon S \to \mathrm{Pow}(S)$ which makes $\mathcal{S}$ regular,
- the relation $\overline{wc} = \{(a,b) \in S \times S \mid a \in wc(b)\}$ is countable,
- for each $(a,b) \in \overline{wc}$, there exists a choice of finitary scales from $\{a\}$ to $\{b\}$.

Lemma. The localic completion $\mathcal{M}(X)$ of a compact metric space X is isomorphic to an overt compact enumerably completely regular formal topology.

Proof. Since X is separable, there exists a countable dense subspace $Y \subseteq X$, and we have $\mathcal{M}(Y) \cong \mathcal{M}(X)$. Moreover, the axiom

$$\textbf{(M1)} \quad \mathsf{b}(x,\varepsilon) \lhd_X \{\mathsf{b}(x',\delta) \in M_X \mid \mathsf{b}(x',\delta) <_X \mathsf{b}(x,\varepsilon)\}$$

of localic completion is equivalent (under the other axiom) to

$$\mathsf{b}(x,\varepsilon) \lhd_X \{\mathsf{b}(x,\varepsilon) \in M_X \mid \delta < \varepsilon\}\,.$$

Define $wc(\mathsf{b}(x,\varepsilon)) \stackrel{\text{def}}{=} \{\mathsf{b}(x,\delta) \in M_X \mid \delta < \varepsilon\}\,.$ □

Point-free characterisation of compact metric spaces

Theorem. Let $\mathcal{S}$ be a formal topology. Then, the following are equivalent:

1. $\mathcal{S}$ is isomorphic to an overt compact enumerably completely regular formal topology.
2. $\mathcal{S}$ is isomorphic to a compact overt subtopology of $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$.
3. $\mathcal{S}$ is isomorphic to a localic completion of some compact metric space.

Proof. (3 ⇒ 1). The previous Lemma.

(1 ⇒ 2). If $\mathcal{S}$ is overt compact enumerably completely regular, then the relation $\overline{wc}$ associated with its function $wc : S \to \mathsf{Pow}(S)$ is countable. Since each $(a,b) \in \overline{wc}$ have a choice of scales, $\overline{wc}$ defines a sequence of maps $\mathcal{S} \to \mathcal{I}[0,1]$, and thus it determines a map $r : \mathcal{S} \to \prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$. Regularity of $\mathcal{S}$ ensures that r is an embedding.

(2 ⇒ 3). If $\mathcal{S}$ is an overt compact subtopology of $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$, then it is a subtopology of $\mathcal{M}(\prod_{n \in \mathbb{N}}[0,1])$. Since $\prod_{n \in \mathbb{N}}[0,1]$ is a compact metric space, $\mathcal{S}$ is isomorphic to a localic completion of some compact metric space. □

Point-free covering theorem ($\mathbf{AC}_\omega$)

Theorem. Any overt compact enumerably completely regular formal topology $\mathcal{S}$ is an image of the Formal Cantor space.

Proof We can identify $\mathcal{S}$ with a compact overt subtopology $\mathcal{S}_{Pos}$ of $\prod_{n \in \mathbb{N}} \mathcal{I}[0,1]$, which is determined by its positivity predicate Pos. Explicitly, $\mathcal{S}_{Pos}$ is generated by the axiom-set on $(S, \leq)$:

$$S = \mathrm{Fin}(\mathbb{N} \times R) \quad \text{where} \quad R = \{(p,q) \in \mathbb{Q} \times \mathbb{Q} \mid p < q\},$$

$$A \leq B \overset{\text{def}}{\iff} \forall\, (n,(p,q)) \in B\, \exists (m,(p',q')) \in A\, [m = n \;\&\; (p',q') \leq (p,q)],$$

$$(p',q') \leq (p,q) \overset{\text{def}}{\iff} p \leq p' \;\&\; q' \leq q,$$

$$(p',q') < (p,q) \overset{\text{def}}{\iff} p < p' \;\&\; q' < q$$

(S1) $S \vartriangleleft_{Pos} \{\{(n,a)\} \in S \mid a \in R\}$ for each $n \in \mathbb{N}$,

(S2) $\{(n,a),(n,b)\} \vartriangleleft_{Pos} \{\{(n,c)\} \in S \mid c \leq_i a \;\&\; c \leq_i b\}$,

(S3) $\{(n,a)\} \vartriangleleft_{Pos} \{\{(n,b)\} \in S \mid b < a\}$

(S4) $S \vartriangleleft_{Pos} \left\{\{(n,(p,q))\} \in S \mid q - p = 2^{-k}\right\}$ for each $n,k \in \mathbb{N}$.

(S5) $A \vartriangleleft_{Pos} \{B \in S \mid B = A \;\&\; B \in Pos\}$.

Point-free covering theorem (AC$_\omega$)

For each $i, k, n \in \mathbb{N}$, define

$$
\begin{aligned}
C_k^i &= \left\{ \{(i,(p,q)\} \in S \mid q - p = 2^{-k} \right\}, \\
D_n &= C_{2^{-n}}^0 \downarrow \cdots \downarrow C_{2^{-n}}^{n-1} \\
&=_{\mathcal{S}_{Pos}} \left\{ \{(0,a_0),\cdots,(n-1,a_{n-1})\} \in S \mid (\forall k < n)\, a_k \in C_n^k \right\}.
\end{aligned}
$$

By (S4) and (S5), we have for all $n \in \mathbb{N}$,

$$S \lhd_{Pos} D_n \cap Pos.$$

Since $\mathcal{S}_{Pos}$ is compact

$$(\forall n \in \mathbb{N})\, (\exists F_n \in \mathrm{Fin}(D_n \cap Pos))\, S \lhd_{Pos} F_n.$$

By **AC**$_\omega$, we obtain a sequence $(F_n)_{n\in\mathbb{N}}$ of finite positive subcovers of $(D_n)_{n\in\mathbb{N}}$. Let $(A_n)_{n\in\mathbb{N}}$ be an enumeration of $\bigcup_{k\in\mathbb{N}} F_k$.

Point-free covering theorem (AC_ω)

The formal Baire space $\mathcal{B} = (B, \lhd_{\mathcal{B}}, \leq)$ is an inductively generated formal topology with base $B = \mathbb{N}^*$ ordered by $l \leq l' \overset{\text{def}}{\iff} l' \preccurlyeq l$ together with the axiom

$$l \lhd_{\mathcal{B}} \{l * \langle n \rangle \mid n \in \mathbb{N}\}.$$

A **fan** on $\mathcal{B}$ is an inhabited decidable subset $T \subseteq B$ such that

1. $l \leq l' \;\&\; l \in T \implies l' \in T$.
2. $l \in T \implies (\exists n \in \mathbb{N})\, l * \langle n \rangle \in T$.
3. $(\forall l \in T)\, (\exists m \in \mathbb{N})\, (\forall n \in \mathbb{N})\, l * \langle n \rangle \in T \implies n \leq m$.

Given an enumeration $(A_n)_{n \in \mathbb{N}}$ of $\bigcup_{k \in \mathbb{N}} F_k$, we can define a fun T by

$$\begin{aligned} T_0 &= \{\langle\rangle\}, \\ T_{n+1} &= \left\{ l * \langle i \rangle \in B \mid l \in T_n \;\&\; A_i \in F_{n+1} \;\&\; A_{l(|l|-1)} \asymp A_i \right\}, \\ T &= \bigcup_{n \in \mathbb{N}} T_n, \end{aligned}$$

$$A \asymp B \overset{\text{def}}{\iff} \forall (n,(p,q)) \in A\, \forall (m,(r,s)) \in B\; [n = m \implies (p,q) \between (r,s)].$$

Since $A_i \lhd_{Pos} F_n \downarrow A_i$ and $A_i \in Pos$ for each $i, n \in \mathbb{N}$, T is a fan.

Point-free covering theorem (AC$_\omega$)

Let $\mathcal{B}_T$ be the (weakly) closed subtopology of $\mathcal{B}$ determined by the fan T, i.e. a subtopology of the formal Baire space determined by the extra axiom

$$l \lhd_T \emptyset \qquad \text{if } l \notin T.$$

Define a formal topology map $r : \mathcal{B}_T \to \mathcal{S}_{Pos}$ by

$$l \, r \, A \overset{\text{def}}{\iff} l \in T \ \& \ \overline{A_{l(|l|-1)}} < A$$

where, letting $k = |l|$

$$\overline{A_{l(k-1)}} \overset{\text{def}}{=} \left\{ (n, (p - 2^{-k}, q + 2^{-k})) \mid (n, (p, q)) \in A_{l(k-1)} \right\},$$

$$A < B \overset{\text{def}}{\iff} \forall (n, b) \in B \exists (m, a) \in A \, [n = m \ \& \ a < b] \, .$$

Point-free covering theorem (AC_ω)

By the standard argument, we can embed $\mathcal{B}_T$ as a fan on the formal Cantor space $\mathcal{C}$:

$$\mathcal{B}_T \overset{s}{\hookrightarrow} \mathcal{C}$$

$$l\, s\, a \overset{\text{def}}{\iff} l \in T \ \&\ a \preccurlyeq \langle 0\,\underbrace{1,\cdots,1}_{l(0)}\rangle * \cdots * \langle 0\,\underbrace{1,\cdots,1}_{l(|l|-1)}\rangle.$$

The image $Im(\mathcal{B}_T)$ is a closed subtopology of $\mathcal{C}$ determined by the fan $Pos = sT = \{a \in 2^* \mid (\exists l \in T)\, l\, s\, a\}$ on $\mathcal{C}$. There exists a surjection.

$$\mathcal{C} \overset{u}{\twoheadrightarrow} Im(\mathcal{B}_T)$$

$$a\, u\, b \overset{\text{def}}{\iff} \widetilde{a} = b$$

where

$$\widetilde{\langle\rangle} = \langle\rangle, \qquad \widetilde{a * \langle i\rangle} = \begin{cases} \widetilde{a} * \langle i\rangle & \text{if } \widetilde{a} * \langle i\rangle \in Pos, \\ \widetilde{a} * \langle j\rangle & j \equiv i+1 \pmod 2 \text{ otherwise.} \end{cases}$$

Corollary. Fan theorem suffices to show that every compact metric space is topologically compact.

References

Thierry Coquand, Giovanni Sambin, Jan Smith, and Silvio Valentini. Inductively generated formal topologies. *Ann. Pure Appl. Logic*, 124(1-3): 71 – 106, 2003.

Thierry Coquand, Erik Palmgren, and Bas Spitters. Metric complements of overt closed sets. *MLQ Math. Log. Q.*, 57(4):373–378, 2011.

Peter T Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

Erik Palmgren. A constructive and functorial embedding of locally compact metric spaces into locales. *Topology Appl.*, 154:1854–1880, 2007.

Giovanni Sambin. Intuitionistic formal spaces — a first communication. In D. Skordev, editor, *Mathematical Logic and its Applications*, volume 305, pages 187–204. Plenum Press, 1987.

Bas Spitters. Locatedness and overt sublocales. *Ann. Pure Appl. Logic*, 162 (1):36–54, 2010.

Steven Vickers. Localic completion of generalized metric spaces I. *Theory Appl. Categ.*, 14(15):328–356, 2005.