Interpretation of set theory into theory of operations

Takako Nemoto (JAIST)
joint work with Hajime Ishihara

Motivation

Aczel defined an interpretation of CZF into Martin-Löf type theory with W-type s.t.

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Notice that we have the operation $\lambda x.x$.
It represents the universal set $U = \{x : x = x\}$!

Theory of operations

The language L_{APP} of APP (EON)

- Variables: x, y, z...
- Constants: k, s (combinators), p, p_0, p_1 (pairing and unpairing operators), 0 (zero), S (successor), P (predecessor), d (definition by numerical cases) and i.
- Binary partial operator: $\circ$
- Predicate constants: $=$, N and E

Notation

Nt and $s \circ t$ are also written $t \in \mathsf{N}$ and st, respectively.

Axioms of APP

- Intuitionisitic logic of partial terms
  - Strictness
    - $t = s \to \mathsf{E}t \wedge \mathsf{E}s$
    - $\mathsf{N}t \to \mathsf{E}t$
    - $\mathsf{E}(ts) \to \mathsf{E}t \wedge \mathsf{E}s$
  - Relation between E and $\exists$
    - $\mathsf{E}t \leftrightarrow \exists x(x = t)$
  - Replacement
    - $\mathsf{N}t \wedge t = s \to \mathsf{N}s$
    - $\mathsf{E}(rs) \wedge t = s \to rt = rs$
    - $\mathsf{E}(tr) \wedge t = s \to tr = sr$
- Nonlogical axioms
  - $\mathsf{E}t \to \mathsf{k}st \simeq s$
  - $\mathsf{E}t \wedge \mathsf{E}t' \to \mathsf{E}stt'$
  - $\mathsf{s}tt't'' \simeq tt''(t't'')$
  - $\mathsf{E}t' \to \mathsf{p}_0(\mathsf{p}tt') \simeq t$
  - $\mathsf{E}t' \to \mathsf{p}_1(\mathsf{p}tt') \simeq t'$
  - $0 \in \mathsf{N}$
  - $t \in \mathsf{N} \to \mathsf{S}t \in \mathsf{N}$
  - $\mathsf{S}n \neq 0$
  - $\mathsf{P}0 = 0$
  - $t \in \mathsf{N} \to \mathsf{P}t \in \mathsf{N}$
  - $\mathsf{P}(\mathsf{S}t) \simeq t$
- $\mathsf{E}t_1 \wedge \mathsf{E}t_2 \wedge t, t' \in \mathsf{N} \wedge t \neq t' \to \mathsf{d}t_1t_2tt = t_1 \wedge \mathsf{d}t_1t_2tt' = t_2$
- $A(0) \wedge \forall x \in \mathsf{N}(A(x) \to A(\mathsf{S}x)) \to \forall x \in \mathsf{N}A(x)$

Additional axioms

APP' is APP plus the following axioms.

1. Domain of successor (S_D): $\forall x(x \notin \mathsf{N} \to \neg \mathsf{E}(\mathsf{S}x))$
2. Strictness of unpairing (p_D):
   $\mathsf{E}(\mathsf{p}_0 x) \vee \mathsf{E}(\mathsf{p}_1 x) \to \exists y \exists z(x = \mathsf{p}yz)$
3. Diagonal identity (i): $\forall xy(\mathsf{E}(\mathsf{i}xy) \leftrightarrow x = y)$

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$\mathsf{APP} + \mathsf{S}_D + \mathsf{p}_D$ proves the existence of the following u, R and d_N:

Totally undefined operator $\forall x \neg \mathsf{E}(\mathsf{u}x)$

Recursor strictly on N $\mathsf{E}t' \to \mathsf{R}tt'0 \simeq t \quad \mathsf{E}(\mathsf{R}tt't'') \to t'' \in \mathsf{N}$

$$t'' \in \mathsf{N} \wedge t'' \neq 0 \to \mathsf{R}tt't'' \simeq t'(\mathsf{r}tt'(\mathsf{P}t''))$$

n-case distinction

$$\mathsf{d}_\mathsf{N} nax = \begin{cases} a_i & \text{if } \bar{\mathsf{i}} = x \wedge i < n \wedge a = (a_0, ..., a_{n-1}), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Model for APP'

PRO, a "standard model" for APP

- Operators are interpreted as (indices) of partial recursive function.
- $st = u$ are interpreted as $\{s\}(t) = u$
- Es are interpreted as $s\downarrow$

The above model is also a model for the extended system APP′.

Lemma
$|\mathsf{APP}'| = |\mathsf{HA}| = |\mathsf{APP}|$

Positive comprehension

Positive formulas φ

$$x = y \mid x \in y \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi(x) \mid \forall x \varphi(x)$$

Consistency of positive comprehension and positive set theory

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- Such set theory was first proposed by Malitz in 1976.
- Very strong variants in which ZF can be interpret are given Esser. (He also gave topological models for them)

Weak positive set theory WPS

Language of set theory

Binary relation $\in$ and $=$ Unary function $\bigcup$ - and -$^{<\omega}$

Constant $\emptyset$ and ω, Binary function $\{\text{-},\text{-}\}$.

Axioms of WPS (based on Intuitionistic logic)

- Empty set
- Pair
- Union
- Infinite set
- Finite sequences: $a^{<\omega} = \{x : x \text{ is a f.s. from } a\}$
- Induction on ω: $A(0) \wedge \forall x \in \omega(A(x) \to A(x')) \to \forall x A(x)$, where $x' = x \cup \{x\}$ and $a \cup b = \bigcup\{a, b\}$.
- WP-comprehension: $\exists a \forall x(x \in a \leftrightarrow \varphi(x))$ for $\varphi \in \mathrm{WP}$, where $\mathrm{WP} \equiv p(\text{atomic}) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi(x) \mid n \in \omega \wedge \forall x \in n \varphi(x)$

Property of WPS

Some theorems in WPS

- Linearity of ω w.r.t. $\in$: $\forall x \in \omega \forall y \in \omega (x \in y \vee x = y \vee y \in x)$
- Existence of Universal set: $U = \{x : x = x\}$
- Primitive recursion on ω: For graphs of $f : U^n \to \omega$ and $g : \omega \times U^n \to \omega$, there is a graph of $h : \omega \times U^n \to \omega$ such that

$$h(n, \vec{a}) = \begin{cases} f(\vec{a}) & \text{if } x = 0 \\ g(h(n, \vec{a}), \vec{a}) & \text{otherwise} \end{cases}$$

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Corollary

HA is interpretable in WPS, therefore $|\mathsf{HA}| \leq |\mathrm{WPS}|$.

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- $((a)^{<\omega})^* \equiv$ $\lambda y.\mathsf{d_N}(\mathsf{p}_0 a)(y(\mathsf{p}_0(\mathsf{p}_1 a)), y(\mathsf{p}_0(\mathsf{p}_1^2 a), ..., y(\mathsf{p}_0(\mathsf{p}_1 a)^{\mathsf{p}_0 a} a)))$

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- $$\begin{array}{ll} (x \in y)^* \equiv \exists z(y = xz) & (x = y)^* \equiv x = y \\ (\neg\psi)^* \equiv \neg\psi^* & (\psi \to \eta)^* \equiv \psi^* \to \eta^* \\ (\psi \wedge \eta)^* \equiv \psi^* \wedge \eta^* & (\psi \vee \eta)^* \equiv \psi^* \vee \eta^* \\ (\forall x\psi(x)) \equiv \forall x\psi(x)^* & (\exists x\psi(x)) \equiv \exists x\psi(x)^* \end{array}$$

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Proof of WP-comprehension

Lemma

For any formula $\varphi(\vec{x})$ in $\in \mathrm{WP}$ with exactly displayed free variables, there is an operator σ_φ in APP' such that

$$\mathsf{APP}' \vdash \forall \vec{a}(\varphi^*(\vec{a}) \leftrightarrow \exists w \mathsf{E}(\sigma_\varphi \vec{a} w)).$$

In particular, if $\varphi(x)$ is in WP with unique free variable x and some parameters, there exists τ_φ such that

$$\mathsf{APP}' \vdash \forall x(\varphi^*(x) \leftrightarrow \exists y \mathsf{E}(\tau_\varphi x y)).$$

Theorem

$\mathsf{APP}' \vdash \psi^*$ for each instance ψ of WP-comprehension.

(Proof) $\lambda x.\mathsf{k}(\mathsf{p}_0 x)\sigma_\varphi(\mathsf{p}_0 x)(\mathsf{p}_1 x) = (\{y : \varphi(y)\})^*$.

Interpretation Theorem

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Corollary

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Further extensions

Class of formulas

- $\mathrm{WP} \equiv p(\text{atomic}) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x \varphi(x) \mid n \in \omega \wedge \forall x \in n \varphi(x)$
- $\mathrm{Pos} \equiv \varphi(\mathrm{WP}) \mid \forall x \varphi(x)$
- $\mathrm{EP} \equiv p(\mathrm{Pos}) \mid \forall x \in a \varphi(x)$

Question

Can we interpret Pos- or EP-comprehension to any system of operations in a similar way?

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Partial answer A variant of Bon(μ) interprets WPS and Pos-comprehension, where Bon(μ) is the system APP with the following μ operation:

$$f : \mathsf{N} \to \mathsf{N} \leftrightarrow \mu f \in \mathsf{N}$$

$$f : \mathsf{N} \to \mathsf{N} \wedge \exists n(fn = 0) \leftrightarrow f(\mu f) = 0$$

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- Find appropriate systems of operations to interpret
  - Pos- and EP-comprehension
  - "usual" set theory with negative comprehension, foundation and extensionality, like CZF
    (It seems some kinds of induction on definedness is needed)

Related works

- Research on systems of operations and explicit mathematics in Bernese group
- Realizability interpretation of CZF into explicit mathemtaics (Tupailo)

References

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