TOPOLOGICAL SEMANTICS FOR VISSER'S PROPOSITIONAL LOGICS

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JAPAN ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY 1990

SOUTHWEST UNIVERSITY

$$\vdash \varphi \quad \Longleftrightarrow \quad \vdash G(\varphi)$$

Intuitionistic Logic

Kripke Models (1965)

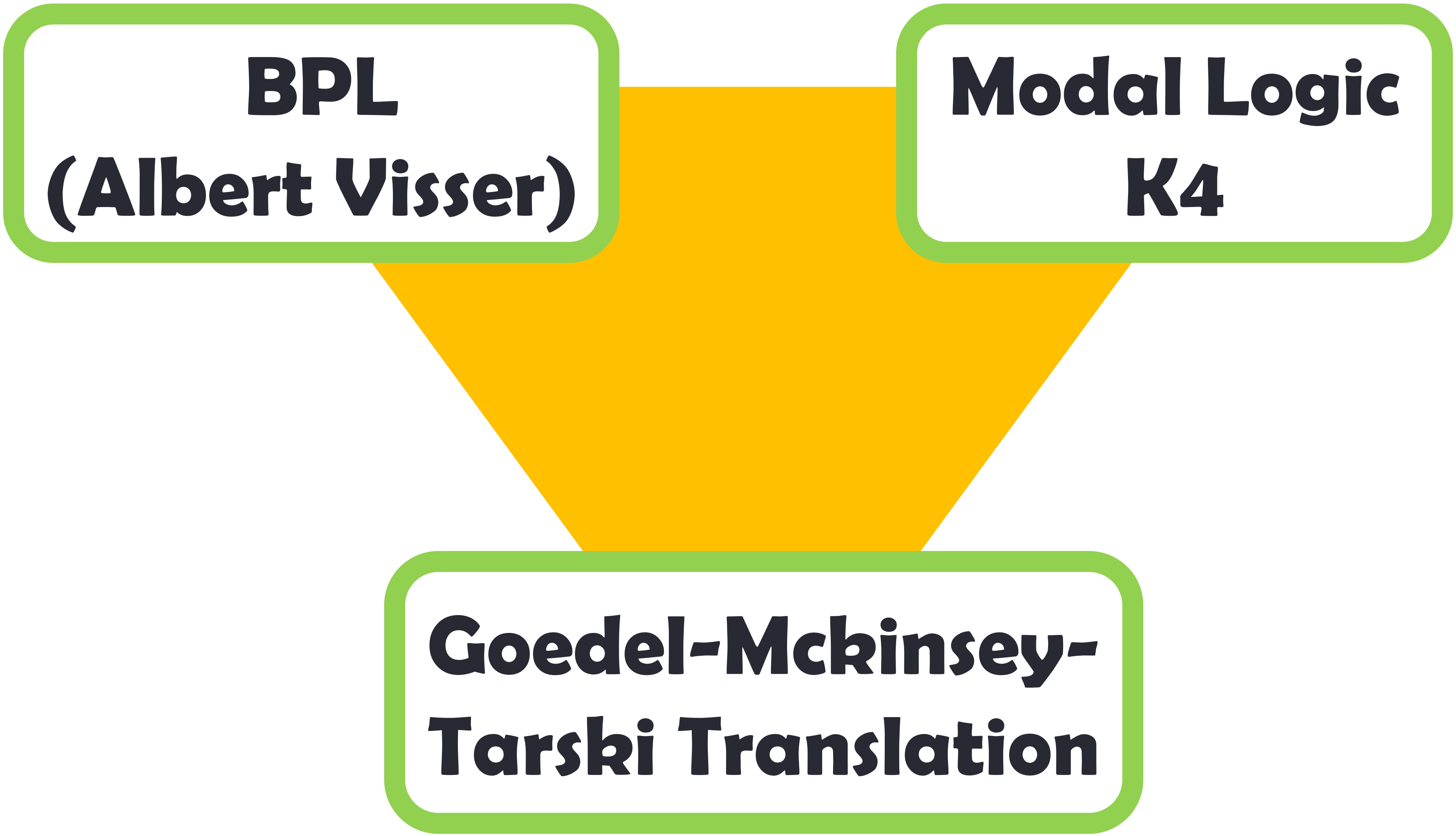
Modal Logic S4

Kripke Models (1959)

Goedel-Mckinsey-Tarski

$$p \mapsto \Box p$$

$$p \to q \mapsto \Box(p \to q)$$


BPL
(Albert Visser)
Modal Logic
K4
Goedel-Mckinsey-
Tarski Translation

$$\vdash \varphi \iff \vdash G'(\varphi)$$

BPL
(Albert Visser)

wK4
(Leo Esakia)

Top. Sem.
(This Talk)

Top. Sem.
(Esakia 2004)

A Variant of
GMT

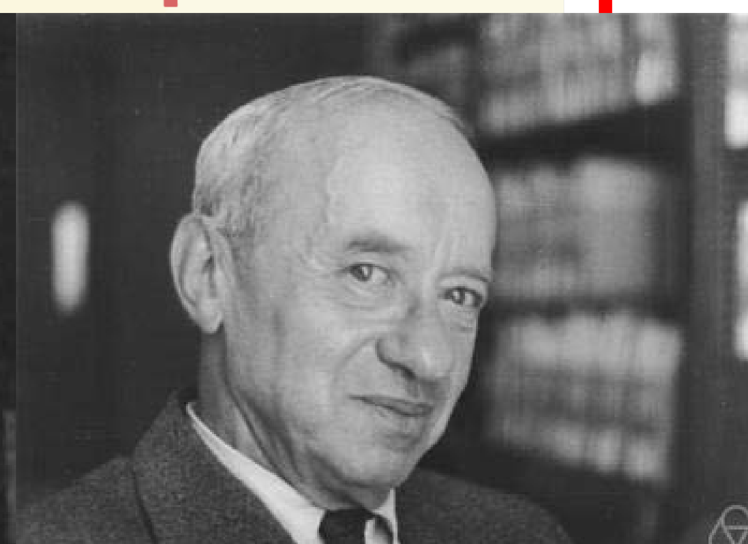
$$p \mapsto p \wedge \Box p$$

$$p \to q \mapsto \Box(p \to q)$$

Thanks to
David Gabelaia
@ Georgia

Top. Sem.

Top. Sem.



Outline of This Talk

Part I

BPL
(Albert Visser)

Part III

wK4
(Leo Esakia)

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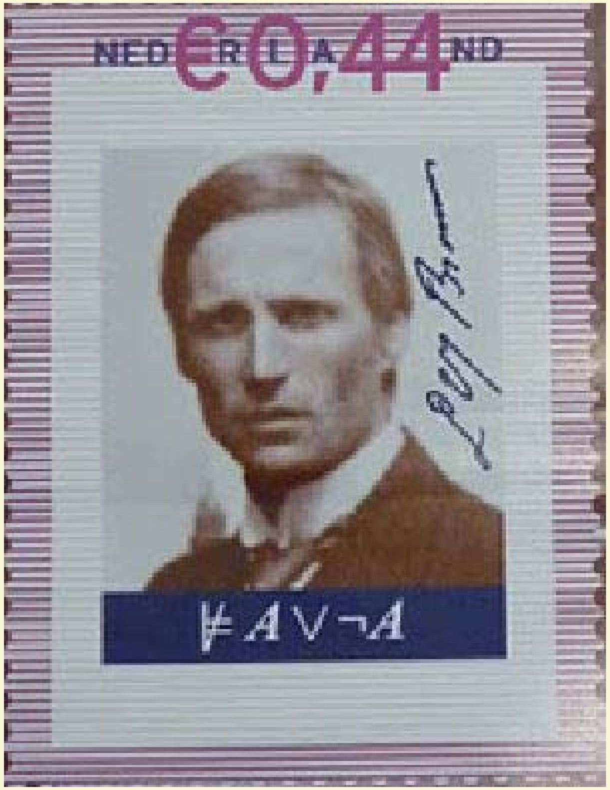
Part II

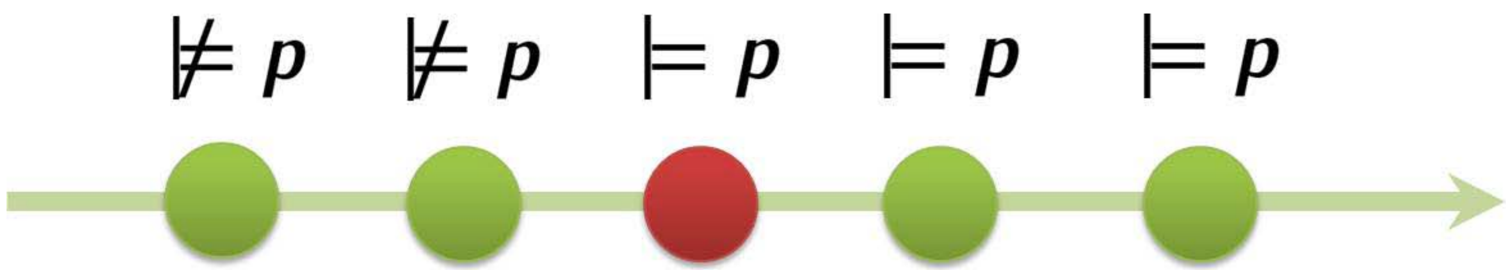
Intuitionist(ic) Logic



Kripke Semantics for IL

- Kripke model for IL: M = (W,R,V)
  - R is a **pre-order** (ref. & tran.)
  - V: Prop -> P(W) is **persistent**:

if w is in V(p) & wRw', then w' is in V(p)



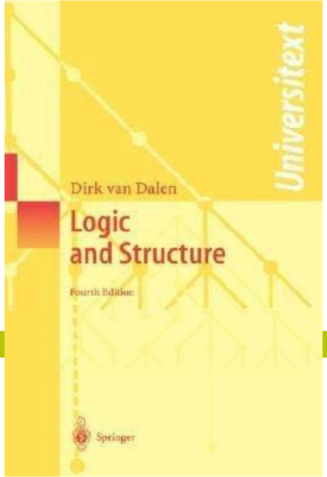
Kripke Semantics for IL (cont.)

$w \models p$	iff	$w \in V(p)$
$w \models \bot$		Never
$w \models \varphi \wedge \psi$	iff	$w \models \varphi$ and $w \models \psi$
$w \models \varphi \vee \psi$	iff	$w \models \varphi$ or $w \models \psi$
$w \models \varphi \rightarrow \psi$	iff	

for all $v \in W$ s.t. $wRv(v \models \varphi \Rightarrow v \models \psi)$

□(φ->ψ), where \`->' is material.

Persistency can be extended to all the formulas of IL (we need transitivity of R).



(Kripke 1965) TFAE:

(1) φ is a theorem of IL

(2) φ is valid on all Kripke models for IL.

Visser's BPL (1981)

We keep the def. of satisfaction |=

- Drop the requirement of ref. of R from Kripke models for IL
- BPL is the set of valid formulas on all transitive Kripke models.
- Logical consequence ($\Gamma \models \varphi$):

for all tran. M and all points w in M,

if all the elements of Γ are true at w,

then φ is also true at w.

Non-Valid Formulas, etc

We can avoid Curry's paradox!

- **all the non-theorems of IL**
- |/= (φ∧(φ->ψ)) -> ψ
- |/= (φ-> (φ->ψ)) -> (φ->ψ)
- **Deduction Theorem fails!**

$$\varphi, \psi \models \theta \Rightarrow \varphi \models \psi \to \theta$$

$$\varphi, \psi \models \theta \not\Leftarrow \varphi \models \psi \to \theta$$

Hilbert-style Axiomatization of BPL: Ono & Suzuki (1997)

- $\varphi \to \varphi$
- $(\varphi \to \psi) \wedge (\psi \to \gamma) \to (\varphi \to \gamma)$
- $\varphi \wedge \psi \to \varphi$
- $\varphi \wedge \psi \to \psi$
- $(\gamma \to \varphi) \wedge (\gamma \to \psi) \to (\gamma \to \varphi \wedge \psi)$
- $\varphi \to \varphi \vee \psi$
- $\psi \to \varphi \vee \psi$
- $(\varphi \to \gamma) \wedge (\psi \to \gamma) \to (\varphi \vee \psi \to \gamma)$
- $\varphi \wedge (\psi \vee \gamma) \to (\varphi \wedge \psi) \vee (\varphi \wedge \gamma)$
- $\bot \to \varphi$
- $\varphi \to (\psi \to \varphi)$
- $\varphi \to (\psi \to \varphi \wedge \psi)$

If $\models \varphi$ and $\models \varphi \to \psi$, then $\models \psi$

valid, even if `->' is a strict implication (Corsi (1987)'s F)

valid because of persistency

Proof Theory of BPL

- Natural Deduction: Visser (1981)
- Hilbert-style Axiomatization
  - Theorems: Ono & Suzuki (1997)
  - Consequence Rel.: Sasaki (1999)
- Genzen-style Sequent Calculus
  - Ishii, Kashima & Kikuchi (2001)

Further Studies on BPL

- **Algebraic Study of BPL :**
  Alizadeh&Ardeshir (2006,2009)
- **FO-extension of BPL :**
  Ruitenburg(1998), Ishigaki&Kikuchi(2007)
- **Naive set theory over BPL:**
  Ruitenburg (1991,1993)
- **Prim. Rec. Realizablity** Plisko (2007)

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A Variant of
GMT-translation

Part II

Goedel-Mckinsey-Tarski Translation

$$G(p) = \Box p$$

$$G(\bot) = \Box \bot$$

$$G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi)$$

$$G(\varphi \vee \psi) = G(\varphi) \vee G(\psi)$$

$$G(\varphi \rightarrow \psi) = \Box(G(\varphi) \rightarrow G(\psi))$$

Known Results on GMT-translation

Goedel, Mckinsey & Tarski

$$\mathbf{Int} \vdash \varphi \iff \mathbf{S4} \vdash G(\varphi)$$

Visser (1981)

$$\mathbf{BPL} \vdash \varphi \iff \mathbf{K4} \vdash G(\varphi)$$

Transforming GMT-translation

$$G(p) = \Box p$$

$$G(\bot) = \Box \bot$$

$$G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi)$$

$$G(\varphi \vee \psi) = G(\varphi) \vee G(\psi)$$

$$G(\varphi \to \psi) = \Box(G(\varphi) \to G(\psi))$$

Transforming GMT-translation

$$G(p) = p \wedge \Box p$$

$$G(\bot) = \Box\bot$$

$$G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi)$$

$$G(\varphi \vee \psi) = G(\varphi) \vee G(\psi)$$

$$G(\varphi \to \psi) = \Box(G(\varphi) \to G(\psi))$$

Transforming GMT-translation

$$G(p) = p \wedge \Box p$$

$$G(\bot) = \bot$$

$$G(\varphi \wedge \psi) = G(\varphi) \wedge G(\psi)$$

$$G(\varphi \vee \psi) = G(\varphi) \vee G(\psi)$$

$$G(\varphi \to \psi) = \Box(G(\varphi) \to G(\psi))$$

$$\mathbf{BPL} \vdash \varphi \iff \mathbf{wK4} \vdash G'(\varphi)$$

$$G'(p) = p \wedge \Box p$$

$$G'(\bot) = \bot$$

Persistency of valuation

$$G'(\varphi \wedge \psi) = G'(\varphi) \wedge G'(\psi)$$

$$G'(\varphi \vee \psi) = G'(\varphi) \vee G'(\psi)$$

``Strict Implication''-like

$$G'(\varphi \to \psi) = \Box(G'(\varphi) \to G'(\psi))$$

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Weak K4 (wK4) by Esakia

- wK4 = K + $(p \wedge \Box p) \to \Box\Box p$
- $(p \wedge \Box p) \to \Box\Box p$ corresponds to: **weak transitivity**, i.e., **if xRy and yRz and x≠z, then xRz**

(Esakia 2001) TFAE:

(1) φ is a theorem of wK4

(2) φ is valid on all weak-tran. models.

Look at proper future points!

Proper Successor Sem. for BPL

- Let M be a Kripke Model for IL
- Change the clause for -> into:

Ordinary Semantics

for all $v \in W$ s.t. $wRv(v \models \varphi \Rightarrow v \models \psi)$

Proper Successor Semantics (New!)

for all $v \in W$ s.t. $w\underline{R}v(v \models \varphi \Rightarrow v \models \psi)$

$$w\underline{R}v \overset{\text{def.}}{\Longleftrightarrow} wRv \text{ and } w \neq v$$

all the valid formulas on all weak tran. models (ord. sem.)

Alternative Kripke sem. for BPL

Sem. ＼ R	Weak tran.	Tran.	Pre-order	Partial Order
Ordinary Sem. for ->	BPL	BPL (Visser)	Int (Kripke)	Int (Kripke)
Proper Suc. Sem. for ->	BPL	BPL	BPL	BPL

Definition of Topological Space

$(W, \tau : W \to \mathcal{P}\mathcal{P}(W))$ is a topological space if:

- If $X \in \tau(w)$ and $X \subseteq Y$ then $Y \in \tau(w)$.
- If $X, Y \in \tau(w)$, then $X \cap Y \in \tau(w)$.
- $W \in \tau(w)$.
- If $X \in \tau(w)$, then $w \in X$
- If $X \in \tau(w)$, then $\{ v \mid X \in \tau(v) \} \in \tau(w)$.

i(X) := { v ∈ W | X ∈ τ(v)}

X is open if X = i(X)

Topological Semantics for Int

$[|\ \varphi\ |] := \{\ x \in W \mid x \models \varphi\ \}$

- **Topological Model: M = (W,τ,V)**
  - **(W,τ) is a topological space**
  - **V: Prop->P(W) is topo-persistent:**

V(p) is open, i.e., V(p) = i(V(p))

$$w \models \varphi \to \psi \quad \text{iff}$$

$$w \in i((W \setminus [|\varphi|]) \cup [|\psi|])$$

Topological Semantics for BPL

- Let M be a Topological Model.
- Change the clause for -> of IL into:

$$w \models \varphi \to \psi \quad \text{iff}$$

$$\exists X \in \tau(w).(X \setminus \{w\}) \cap [|\varphi|] \subseteq [|\psi|]$$

$$\textbf{iff} \quad w \in t((W \setminus [|\varphi|]) \cup [|\psi|])$$

Eliminate the current point

(co-)Derived-set Operation

$x \in d(X)$ iff x in the closure of $X-\{x\}$.

$$f'(x) = \lim_{x \to a, x \neq a} \frac{f(x) - f(a)}{x - a}$$

- $t(X) := (d(X^c))^c$
  where $d(X)$ is the **derived-set op.**
- '$a \in d(X)$' in the real line means

there exists a sequence $(a_n \mid n \in N)$ s.t. $a \neq a_n \in X$ and $\lim_{n \to \infty} a_n = a$

Topological Sem. for BPL

``T-zero'' space

Sem. \ Sp.	Top. Spaces	T0-Spaces
Interior op.	Int	Int
Co-derived set op.	BPL	BPL

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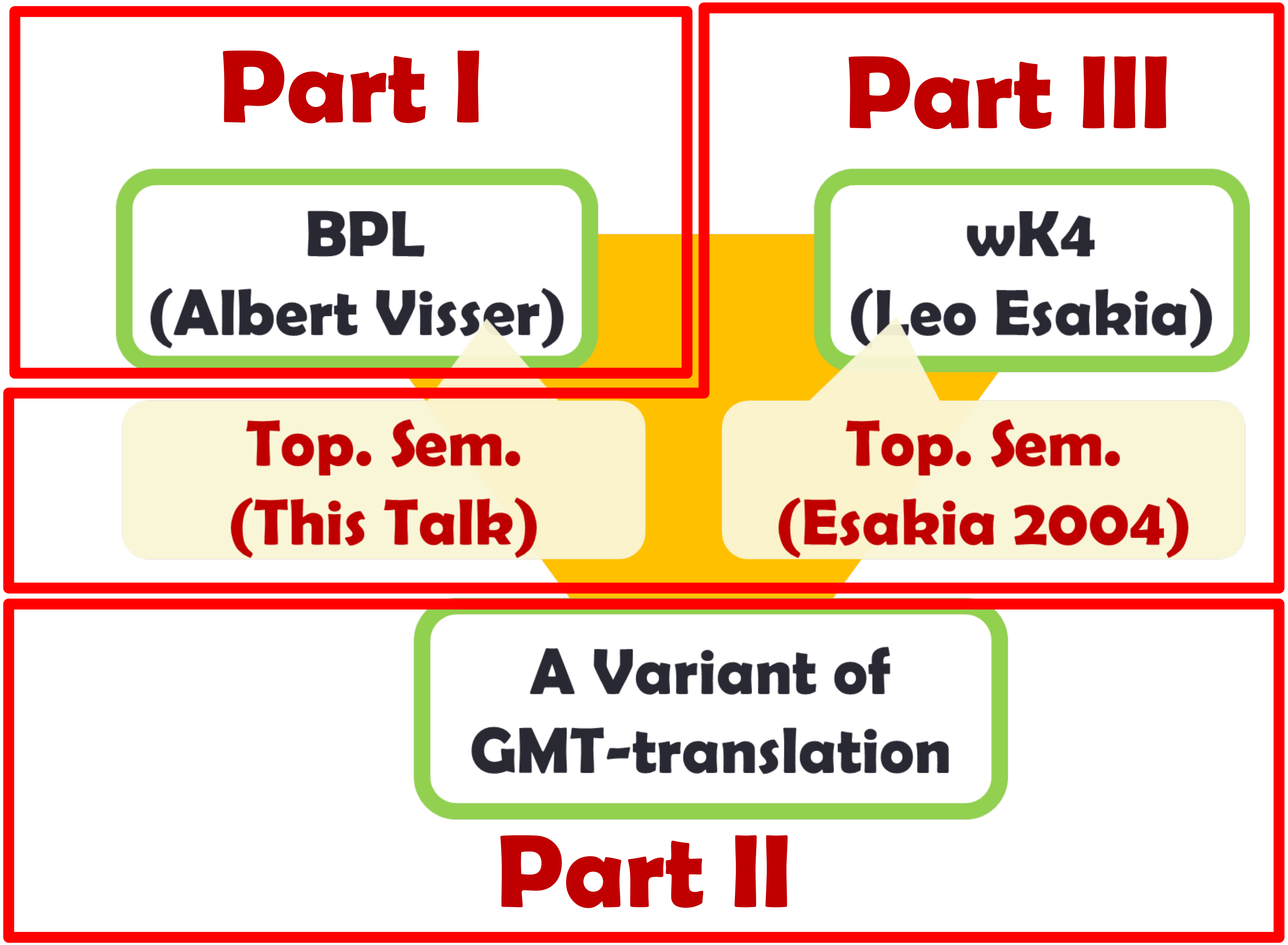

Further Directions

- Dual BPL w/ Top. Sem. (Dual Int)
- Top. Semantics for FO-B(P)L
- Constructive Proof of GMT trans.
  - from Int to S4 (Negri and Von Plato)
  - from BPL to K4 (Yamazaki and S.)
- Embedding from Int to BPL.

  (p.c. from Minghui Ma and Lin Zhe)

Thanks for your attention ☺

Formal Provability Logic (FPL)

- BPL + ((T -> φ)-> φ)->(T-> φ)
- Added one is not a tautology

□(□p->p) -> □p

Visser (1981)

$$\mathsf{FPL} \vdash \varphi \iff \mathsf{GL} \vdash G(\varphi)$$

(Visser 1981) TFAE:

(1) φ is a theorem of FPL

(2) φ is valid on all finite irref. & tran. models.